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A detailed investigation of the spectral rigidity for integrable Hamiltonian systems

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Abstract. We carry out a detailed analysis of the spectral rigidity for integrable systems with a view towards understanding the deviations from the universal behaviour ($L/15$) in systems with degeneracies in orbit actions. In the process, we also investigate the saturation region. Our studies show that the energy dependence of $\bar{\Delta}_3(\infty)$ is a better indicator of the underlying universality in integrable systems.

1. Introduction

There is strong evidence to suggest that the fluctuation properties of the spectra reflect the underlying dynamics of the corresponding classical system. The simplest characteristic of the structure of the energy spectrum is the nearest-neighbour level spacing distribution (hereafter referred to as NNLD) $P(s)$. It was argued by Berry and Tabor [1] that for generic integrable systems where the energy contours in action space are curved, $P(s)$ is given by the Poisson distribution e^{-s}. The levels therefore tend to cluster as the behaviour of $P(s)$ for small spacings suggests. On the other hand, time reversal invariant chaotic Hamiltonian systems are characterized by a level repulsion. The numerical study of Bohigas *et al* [2] strongly indicates that the NNLD of such systems follows the Wigner distribution, a result known to be true for complex systems having many degrees of freedom. Extensive studies [3–5] on several two-dimensional stochastic systems have since confirmed this belief.

Higher-order correlations however contain more information. Among the useful ones is the Δ_3 statistic of Dyson and Mehta [6], often referred to as the spectral rigidity. It is defined as the average of the mean square deviation of the integrated density of states, $N(E)$, from the best fitting straight line, $a + b\varepsilon$, in an interval $[x - L/2, x + L/2]$, where L is the length of the interval along the spectrum in units of the average level spacing. The averaging is over the spectrum or ensemble or a combination of both. It is the first spectral measure for which a theory based on Gutzwiller's periodic orbit sum rule [7], was put forward [8]. According to Berry [8], $\bar{\Delta}_3(L)$ should display a universal behaviour for $L \ll L_{\max}$ ($= h\langle d\rangle / T_{\min}$, where $T_{\min}$ is the period of the shortest orbit and $\langle d\rangle$ is the average density of states) as a consequence of the properties of long periodic orbits. Within this framework, the following universal behaviour is proposed: for generic integrable systems, $\bar{\Delta}_3(L) = L/15$ while for chaotic systems with time reversal symmetry, $\bar{\Delta}_3(L) = \ln(L)/\pi^2 - 0.006\,95$. In the latter case, the universality is valid in the region $1 \ll L \ll L_{\max}$. For values of $L \gg L_{\max}$, $\bar{\Delta}_3(L)$ saturates non-universally. The averaging in this case is over the spectrum.

Most systems in nature however do not belong to these extreme categories. For an N-dimensional generic Hamiltonian system, the phase space is mixed. Berry and Robnik [9] suggest that the spectral fluctuations should result from independently superposing a Poisson spectrum with relative weight ν and a series of GOE spectra corresponding to disconnected chaotic regions of phase space with relative weight $\bar{\nu}_i$ ($\nu + \Sigma\bar{\nu}_i = 1$) where ν and $\bar{\nu}_i$ are assumed to be proportional to the Liouville measure of the regular and chaotic regions of phase space. Their claims have been subjected to several verifications with mixed results. For nearly integrable systems, (ν small) the fit is not so good [10, 11] and the more heuristic Brody distribution [12] is preferred.

In recent years however, the universal properties of the eigenvalue spectrum of generic integrable systems has come under close scrutiny. There are reasons to believe [13] that a system specific character shows up in the spectral statistic (at least NNLD) of these systems and hence it is imperative to reconsider what is universal in the property of eigenvalue sequences of quantum systems that are classically integrable. This naturally has important repercussions on the spectral statistics of nearly integrable systems and might lead to a better understanding of the deviation from the Berry–Robnik distribution mentioned earlier.

In the following, we study the spectral rigidity, Δ_3, for integrable systems. We show that for systems where degeneracies in orbit actions (not necessarily due to symmetries) exist, departures from the $L/15$ behaviour are to be expected as can be explained on the basis of the periodic orbit theory of Gutzwiller. Moreover, since these deviations are strongly system dependent, we investigate the saturation region ($L \gg L_{\max}$) and show that there exists a fundamental property that can be used to characterize integrable and chaotic systems. We support these results with extensive numerical studies on billiard systems. We also make a few remarks about the spectral measures in chaotic systems with degeneracies in periodic orbit actions.

The paper is organized along the following lines. In section 2 we review some of the previous results on the nearest-neighbour-level statistics of integrable systems and arrive at an appropriate form for the spectral rigidity in case of integrable systems with degeneracies in orbit actions. We substantiate our results with numerical studies on billiard systems. In section 3, we show that the energy dependence of its saturation value, $\bar{\Delta}_3(\infty)$, can be used as a characteristic to distinguish between integrable and chaotic systems. Our numerical results for the Bunimovich stadium billiard and rectangular billiards are discussed in the following section. Finally we discuss our results and make some pertinent remarks in the concluding section.

2. Spectral statistics of integrable systems

The integrated density of states, $N(E)$, of a quantum system consists of two parts. Superposed on the average smooth part (the Thomas–Fermi term) is the non-analytic fluctuating contribution. Thus

$$N(E) = N_{\rm av}(E) + N_{\rm fl}(E). \tag{2.1}$$

The first step in an analysis of the spectral fluctuations is to remove the average trend in order to characterize and compare the fluctuations of different systems whose corresponding average behaviours are not the same. This process, known as 'unfolding', can be achieved through the mapping

$$\varepsilon_i = N_{\rm av}(E_i). \tag{2.2}$$

The sequence ε_i thus obtained has a mean spacing equal to unity independent of the particular form of the function N_{av}.

The simplest statistical measure of this new sequence is the nearest-neighbour spacing distribution, $P(s)$. It is defined such that $P(s)\,ds$ = probability of finding adjacent points $(j, j+1)$ with a spacing $(\varepsilon_{j+1}-\varepsilon_j)$ lying between s and $s+ds$.

Berry and Tabor [1] give strong arguments to show that in generic integrable systems, where the energy contours in action space are curved, the levels are uncorrelated and $P(s)$ is simply the negative exponential, e^{-s}, characteristic of a Poisson process. The simplest example of a non-generic case is the two-dimensional harmonic oscillator where the distribution of spacing is strongly dependent on the number theoretic properties of the frequencies.

The Poisson result has been subjected to several verifications but its universal nature has remained in doubt. The first deviations were observed by Berry and Tabor [1] for rectangular billiard systems where the results hold only when $\alpha = a^2/b^2$ is irrational. The difference in behaviour is attributed to the fact that the classical orbits picked out by the quantum condition are all closed when α is rational while they are never closed when it is irrational. The former case is then referred to as non-generic (thus we are forced to redefine what genericity means). While discussing the Δ_3 statistic however we shall show that the explanation tendered is inconsistent.

Recently Shudo [13] has examined the dependence of the arithmetic nature of α in the NNLD for (i) the rectangular billiard having energy eigenvalues

$$E_{m,n} = m^2 + \alpha n^2 \tag{2.3}$$

and (ii) the integrable version of the Morse oscillator system with

$$E_{m,n} = \{2(m+\tfrac{1}{2}) - (m+\tfrac{1}{2})^2\} + \{2(n+\tfrac{1}{2})/\alpha - (n+\tfrac{1}{2})^2/\alpha^2\}. \tag{2.4}$$

Using the continued fraction expansion of the golden mean $(\sqrt{5}-1)/2$, he has studied the frequency of degeneracies and the NNLD for various convergents of α with the following conclusions.

(i) The ratio of degeneracies is much higher for the billiard system. Also no degeneracies were observed above the 15th convergent in the billiard system while they disappeared around the 9th convergent in the Morse oscillator system.

(ii) For the lower convergents considered (3rd and 6th), the level spacing distribution shows a system specific nature and strongly reflects the formula of the eigenvalue sequence.

(iii) Though the NNLD gradually approaches the Poisson distribution for successively higher convergents in both cases, the previous observations suggest that no limiting distribution exists.

Thus even if energy contours form a curved surface, different types of distributions appear in accordance with the arithmetic nature of α.

The system specific behaviour of the NNLD naturally leads to the question whether a similar behaviour persists in higher order correlations as well. Amongst the well studied ones, is the spectral rigidity Δ_3 defined as

$$\bar{\Delta}_3(L) = \left\langle \min_{a,b} \frac{1}{L} \int_{-L/2}^{L/2} d\varepsilon [N(x+\varepsilon) - a - b\varepsilon]^2 \right\rangle \tag{2.5}$$

where the $\langle\,\rangle$ denotes an average over the spectrum or ensemble or a combination of both.

The semiclassical spectral density, $d(E)$ can be written as

$$d(E) = \langle d(E) \rangle + d_{osc}(E) \tag{2.6}$$

where $d_{osc}(E)$ is a sum over classical periodic orbits each of which given an oscillatory contribution. According to Gutzwiller

$$d_{osc}(E) = (1/\hbar^{\mu+1}) \sum A_j(E) \exp(iS_j(E)/\hbar) \tag{2.7}$$

where $\mu = (N-1)/2$ for integrable systems, N being the degrees of freedom. Using (2.7) and (2.5) one arrives at a semiclassical expression for the spectral rigidity [8]

$$\bar{\Delta}_3(L) = \langle (1/\hbar^{2\mu}) \sum\sum (A_i A_j / T_i T_j) \exp\{i(S_i - S_j)/\hbar\} \times [F(y_i - y_j) - F(y_i)F(y_j) - 3F'(y_i)F'(y_j)] \rangle \tag{2.8}$$

where

$$y_j = LT_j/2\hbar\langle d\rangle \qquad F(y) = \sin(y)/y \qquad T_j = dS_j/dE \tag{2.9}$$

and the averaging is over an interval that is large compared with L_{max} but small compared with E. An important ingredient at this juncture, which leads to the $L/15$ behaviour of Δ_3, is the assumption that in the case of integrable systems, the process of averaging washes out the off-diagonal terms and hence the expression for the sum of orbit amplitudes arrived at by Berry and Tabor [1] and subsequently by Hannay and Ozorio de Almeida [14] can be used for $L \ll L_{max}$. For systems with degeneracies in orbit actions, however, off-diagonal terms contribute substantially as we shall now show.

For integrable systems, (2.8) can be written as

$$\bar{\Delta}_3(L) = (2/\hbar^{N-1}) \int (dT/T^2)\phi(T)G(LT/2\langle d\rangle\hbar) \tag{2.10}$$

where

$$\phi(T) = \left\langle \sum_i \sum_j{}^{+} A_{M_i} A_{M_j} \cos\{(S_{M_i} - S_{M_j})/\hbar\}\delta(T - \{T_{M_i} + T_{M_j}\}/2) \right\rangle \tag{2.11}$$

$$G(y) = 1 - \sin^2(y)/y^2 - 3(y\cos(y) - \sin(y))^2/y^4 \tag{2.12}$$

and $\boldsymbol{M}_k$ denotes the winding number $\{M_{1k}, M_{1k}, \ldots, M_{Nk}\}$ around the N-torus. The $^+$ sign on the summations denotes a restriction to positive traversals only.

In order to fix our ideas, we consider an orbit action which is threefold degenerate. Thus there exist three distinct vectors, say $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$ such that the orbits have identical periods and hence equal amplitudes as well. As a consequence, there would be nine terms in $\phi(T)$, each of which would contribute equally. However, if only the diagonal terms in $\phi(T)$ are considered, six of the nine terms would be dropped, leading to the erroneous conclusion that $\bar{\Delta}_3(L) = L/15$. The off-diagonal terms of degenerate orbits can however be incorporated in the diagonal sum

$$\phi_D(T) = \sum A_M^2 \delta(T - T_M) \tag{2.13}$$

by multiplying each term by the degree of degeneracy. Thus, if $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$ are such that the periodic orbits are degenerate, their contribution to $\phi(T)$ can be written as

$$3[A_{M_1}^2\delta(T - T_{M_1}) + A_{M_2}^2\delta(T - T_{M_2}) + A_{M_3}^2\delta(T - T_{M_3})]$$

since each term contributes equally. In the general case of n-fold degeneracy the factor 3 is replaced by n. Thus

$$\phi(T) = \sum n_i A_{M_i}^2 \delta(T - T_{M_i}). \tag{2.14}$$

Not all n_i's are equal however and in general it would be erroneous to replace $\phi(T)$ by $n_{av}\phi_D(T)$ since the average local degeneracy varies with the length of the periodic orbit. However, since the orbit selection function, $G(y)$, picks the periodic orbits which contribute to $\bar{\Delta}_3(L)$, one can, to a first approximation, take the local average of their degeneracies for each value of L. In the region $L \ll L_{max}$, $G(y)$ ensures that only long periodic orbits contribute and hence the asymptotic result of Berry and Tabor [1] for the sum of orbit amplitudes

$$\phi_D(T) \to (d\Omega/dE)/(2\pi)^{N+1} \tag{2.15}$$

can be used. Thus for degenerate systems,

$$\bar{\Delta}_3(L) = n_{av}(L)L/15 \tag{2.16}$$

where $n_{av}(L)$ denotes the average degeneracy of those periodic orbits which contribute to the spectral rigidity at a given value of L.

In order to verify these results, we have considered rectangular billiards with rational α. The energy eigenvalues are given by (2.3). The orbit periods T_M can be expressed as

$$\begin{aligned} T_M &= [2m(M_{1i}^2 a^2 + M_{2i}^2 b^2)/E]^{1/2} \\ &= b[2m(M_{1i}^2\alpha + M_{2i}^2)/E]^{1/2} \end{aligned} \tag{2.17}$$

where a and b denote the lengths of the two sides and $\alpha = a^2/b^2$. The arithmetic properties of the orbit periods and energy eigenvalues are hence identical for the rectangular billiard. For sufficiently low approximants of irrational α, degeneracies abound and thus the system is ideally suited for the verification of our results.

We have computed the $\bar{\Delta}_3$ statistic for three approximants of $\alpha = e$, namely (i) $\alpha = 3$, (ii) $\alpha = 8/3$ and (iii) $\alpha = 109\,601/40\,320$. The levels obtained from (2.3) have been unfolded using the Weyl formula for the integrated density of states. The new sequence of levels obtained through the mapping $\varepsilon_i = N_{av}(E_i)$ have a mean spacing unity. The averaging in all three cases is in an interval $[\varepsilon - \Delta\varepsilon, \varepsilon + \Delta\varepsilon]$ where $\varepsilon = 20\,000$ and $\Delta\varepsilon = 1000$. The interval is sufficiently large to kill the oscillatory off-diagonal terms.

Figure 1 shows plots of $\bar{\Delta}_3(L)$ for the three values of α mentioned above. The full line denotes the $L/15$ behaviour. The curves 1 and 2 start off initially with a slope greater than 1/15 but flatten out gradually indicating a gradual decrease in the value of $n_{av}(L)$. However since the degeneracies in the second case are lower, curve 2 lies closer to the full line. Curve 3 shows the spectral rigidity for the 9th approximant of $\alpha = e$. Since the degeneracies in orbit actions are fewer still, $\bar{\Delta}_3(L)$ approximates $L/15$ quite well.

In order to get a better idea of the underlying phenomenon responsible for the change of slope observed above, we have computed the mean degeneracy, n_{av}, as a function of L. The orbits considered for the averaging at a particular value of L is picked by the selection function $G(LT/2\hbar\langle d\rangle)$. Our results for the three values of α considered above are shown in figure 2. While curve 3 remains steady around 1.0, the average degeneracy in the first two cases gradually decreases with L. This is in keeping with the behaviour observed in figure 1.

The correspondence between the nearest-neighbour-level statistics and the spectral rigidity is apparent in integrable billiard systems. This is due to the fact that the arithmetic properties of eigenvalue sequences and orbit periods are identical. For separable integrable systems with smooth potentials of the form

$$V^{(2m)}(x_1, x_2) = c_1 x_1^{(2m)}/2 + c_2 x_2^{(2m)}/2 \tag{2.18}$$

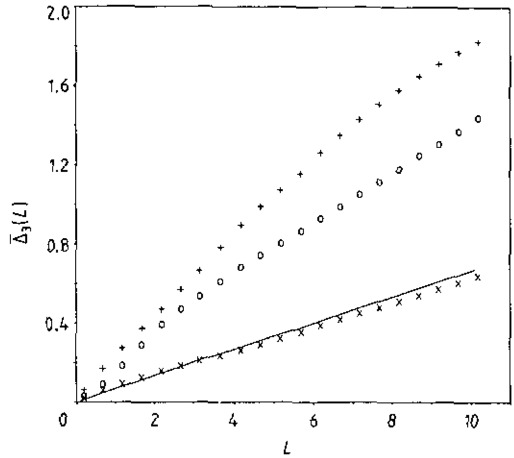

Figure 1. The spectral rigidity, $\bar{\Delta}_3$, for the (1) second (+++) (2) fourth (OOO) and (3) ninth (×××) approximant of $\alpha = e$. The solid line denotes the $L/15$ behaviour. Curves 1 and 2 start with slopes that are high initially but register a gradual decrease. Curve 3 however approximates the Poisson behaviour quite well.

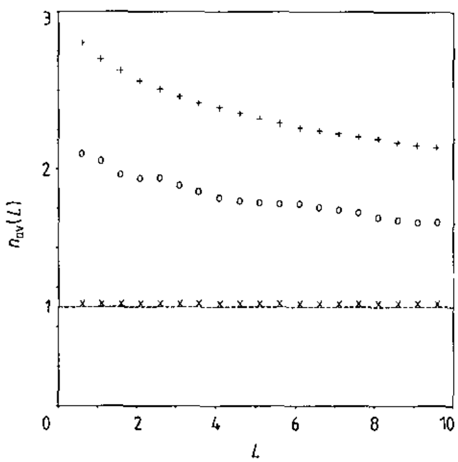

Figure 2. The mean degeneracy, n_{av}, of the periodic orbits as a function of L. The orbits considered at each L value is decided by $G(y)$. Curves 1, 2 and 3 are as in figure 1. While curve 3 remains steady around 1, the average degeneracy for the 2nd and 4th approximant decreases gradually with L.

the energy eigenvalues are given by

$$E_M = \gamma_1(M_1+\alpha_1)^p + \gamma_2(M_2+\alpha_2)^p \tag{2.19}$$

where $p = (2m)/(m+1)$ and lies between 1 and 2. The factors γ_1 and γ_2 depend on c_1 and c_2 while α_1, α_2 are the Maslov indices. For rectangular billiards, $p = 2$. The orbit periods on the other hand can be expressed as

$$T_M = 2\pi E^{1/p}[(M_1/\gamma_1)^{p/p-1} + (M_2/\gamma_2)^{p/p-1}]^{p-1/p}. \tag{2.20}$$

The exponents of the M_i are different for the energy eigenvalues and orbit periods in general and hence their arithmetic properties differ. With a judicious choice of p, γ_1

and γ_2, it is possible to have T_M's of the form

$$T_M = 2\pi E^{n-1/n}\gamma_1^{n-1}[(M_1)^n + l(M_2)^n]^{1/n} \tag{2.21}$$

where n and l are integers. The corresponding expression for energy eigenvalues is

$$E_M = \gamma_2[(M_1+\alpha_1)^{n/n-1}l^{1/n-1} + (M_2+\alpha_2)^{n/n-1}]. \tag{2.22}$$

Though equation (2.21) looks simple enough, not much is known about their arithmetic properties for $n > 2$. However, the above expressions do suggest that degeneracies in orbit periods are higher than in case of energy eigenvalues and hence a situation may arise where the NNLD closely approximates a Poisson distribution but the spectral rigidity shows deviations from $L/15$. We have investigated systems with n equal to 3 and 4. The values of l chosen were 1 and 2 in both cases. While the nearest-neighbour-level statistics fitted well to a Poisson in all four cases, the deviations for the Δ_3 statistic, though visibly different for l equal to 1 and 2, were small. A similar behaviour is seen for the number variance. This suggests that though individual orbits may be degenerate, the mean degeneracy remains close to 1 for systems with $1 < p < 2$.

Finally, we investigate the limit $\hbar \to 0$. It is clear from our analysis that off-diagonal terms arising from degenerate orbits survive in the semiclassical limit. Moreover, with increasing energies, the selection function, $G(y)$, picks orbits of longer lengths at a given value of L. Since their average degeneracies are higher, the deviations get more pronounced in the semiclassical limit.

A similar analysis holds for the number variance and nearest-neighbour spacing distribution as well where terms identical to $\phi(T)$ occur. The off-diagonal terms are necessary to account for the deviations. (In the case of the nearest-neighbour spacing distribution, (3.11) in [1] has to be replaced by the more general expression for $\Pi(K)$ (essentially the same as $\phi(T)$) which includes the off-diagonal terms as well). Deviations for the Poisson behaviour are hence to be expected in systems with degeneracies in orbit actions.

While we have restricted ourselves to integrable systems in the present work, we would like to remark that these ideas apply for chaotic systems as well. For systems with degeneracies in the periodic orbit actions, the diagonal approximation has to be suitably modified (in the region where it holds; for larger lengths off-diagonal terms are necessarily included and the semiclassical sum rule of Berry [8] has to be used) to account for the off-diagonal terms arising from degenerate periodic orbits. Thus deviations from their known universal behaviour are to be expected. A useful example is the (symmetric) Hadamard–Gutzwiller model discussed by Aurich and Steiner [15]. There are very strong degeneracies amongst the periodic orbits and deviations in both the NNLD and spectral rigidity are observed. A similar situation arises in the hyperbola billiard as well (figure 16 in [16]).

The preceding analysis therefore conclusively shows that degeneracies in orbit periods give rise to deviations in the spectral rigidity, Δ_3 from its known 'universal' behaviour. These survive at higher energies and moreover become enhanced. It is easy to verify that similar system-specific behaviour occur in the number variance, Σ^2 as well.

On the other hand, it is expected that integrable systems should possess some universality which distinguishes them from non-integrable systems in general. With this in mind, we investigate the region $L \gg L_{\max}$ in the following section and show that the energy dependence of $\bar{\Delta}_3(\infty)$ can indeed be used as a universal characteristic, applicable to all cases where the energy contours in action space are curved (generic).

3. Energy dependence of $\bar{\Delta}_3(\infty)$ for integrable and chaotic systems

An important fallout of Berry's semiclassical derivation of the spectral rigidity was the prediction of the non-universal saturation region for values of $L \gg L_{\max}$. It follows as a direct consequence of the saturation of the orbit selection function, $G(y)$, mentioned earlier (for a plot of $G(y)$, see [8]). Such a behaviour had been observed earlier for integrable billiard systems [17] and has been verified since for separable systems in general [18]. Recently Aurich and Steiner [5] have been able to observe the saturation phenomenon in the Hadamard–Gutzwiller ensemble.

As in the region $L \ll L_{\max}$, degeneracies in the actions of short periodic orbits affect the magnitude of the saturation value, $\bar{\Delta}_3(\infty)$. Thus, for the square billiard ($\alpha = 1$), the saturation value at a given energy is nearly double that for rectangular billiards with α close to 1. The appropriate expression for $\bar{\Delta}_3(\infty)$ in case of integrable systems is thus:

$$\bar{\Delta}_3(\infty) = (2/\hbar^{N-1}) \sum_i n_i A_{M_i}^2 / T_{M_i}^2 \tag{3.1}$$

where the sum is over all periodic orbits. Since the orbit parameters A_M and T_M are strongly system dependent, the saturation value itself is non-universal. Its energy dependence however contains vital information about the nature of the periodic orbits in the system and hence can be used to characterize integrable (periodic orbits occurring in $(N-1)$-parameter families) and chaotic systems. It is implicitly assumed in the following that all periodic orbits in the latter case are isolated. The occurrence of $(N-1)$-parameter family of periodic orbits in chaotic systems leads to certain consequences which will be discussed in the following section.

For N-dimensional integrable systems with separable potentials of the form

$$V^{(2m)}(x_1 \ldots x_N) = \sum_i C_i x_i^{(2m)} \tag{3.2}$$

the energy dependence of the spectral rigidity for $L \gg L_{\max}$ is given by [17]

$$\bar{\Delta}_3(\infty) \sim (N(E))^{(N-1)/N} \tag{3.3}$$

where $N(E)$ is the integrated density of states at an energy E. The mapping $\varepsilon_i = N(E_i)$ allows us to write (3.3) as

$$\bar{\Delta}_3(\infty) \sim \varepsilon^{(N-1)/N} \tag{3.4}$$

where ε is the unfolded energy of the system. For two-dimensional systems therefore

$$\bar{\Delta}_3(\infty) \sim \varepsilon^{1/2}. \tag{3.5}$$

The result holds for all generic integrable systems.

For chaotic systems with time reversal symmetry, the behaviour is totally different. As in the integrable case, there exists a saturation region for $L \gg L_{\max}$ and the predominant contribution comes from the short orbits. The averaging procedure does not, however, eliminate all the off-diagonal terms. For $L \gg L_{\max}$, the rigidity saturates non-universally at a value approximately given by [8]

$$\bar{\Delta}_3(\infty) \approx \ln(eL_{\max})/\pi^2 - 0.125 \tag{3.6}$$

where $L_{\max} = 2\pi/T_{\min}$ (we have put $\hbar^2/2m = \langle d \rangle = 1$). Thus for billiard systems,

$$\bar{\Delta}_3(\infty) \sim \ln(4\pi e\sqrt{E}/l_0)/\pi^2 - 0.125 \tag{3.7}$$

where l_0 is the corresponding length of the shortest periodic orbit. Thus the energy dependence of the rigidity can be expressed as

$$\bar{\Delta}_3(\infty) \sim \ln(E)^{1/2\pi^2} + C \tag{3.8}$$

where C is a constant independent of energy.

In terms of the unfolded energy

$$\bar{\Delta}_3(\infty) \sim \ln(N^{-1}(\varepsilon))^{1/2\pi^2} + C. \tag{3.9}$$

For two-dimensional chaotic billiards

$$\bar{\Delta}_3(\infty) \sim \ln(\varepsilon)^{1/2\pi^2} + D \tag{3.10}$$

where D is a constant independent of energy.

The characteristic that we wish to investigate, rests on the formulae given by (3.5) and (3.10). In other words, the energy dependence of $\bar{\Delta}_3(\infty)$, can indeed be used to distinguish integrable systems from the chaotic ones. Thus while in case of generic integrable systems the spectral rigidity for $L \gg L_{\max}$ is proportional to $\varepsilon^{1/2}$, it has a logarithmic dependence on the unfolded energy for chaotic systems with isolated unstable periodic orbits. We support these claims numerically in the following section.

4. Numerical results and discussions

In the following, we present our numerical results for billiard systems. As representatives of integrable systems, we have considered the rectangular billiard with (1) $\alpha = 1$ and (2) $\alpha = 8/5$. We have computed $\bar{\Delta}_3(\infty)$ at $L = 120$. It is easy to verify that the spectral rigidity indeed saturates at this value for the considered range of energy. The averaging has been carried out in an interval $[\varepsilon - \delta\varepsilon, \varepsilon + \delta\varepsilon]$, where $\delta\varepsilon$ has been chosen to be 150 and ε is the unfolded energy at which $\bar{\Delta}_3(\infty)$ is calculated. Our results are shown in figure 3. The linear variation with $\sqrt{\varepsilon}$ is obvious in both cases. The fluctuations at

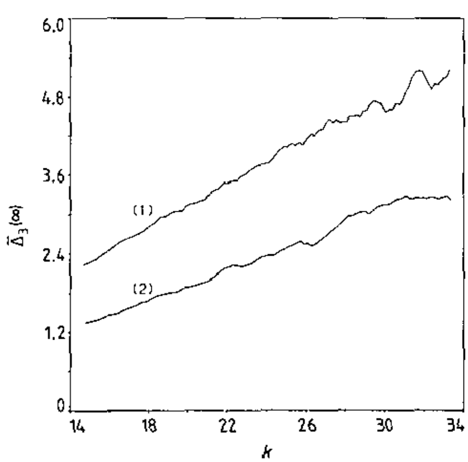

Figure 3. The saturation value of the spectral rigidity $\bar{\Delta}_3(\infty)$ for the rectangular billiard with (1) $\alpha = 1$ and (2) $\alpha = 8/5$, plotted as a function of $k = \sqrt{\varepsilon}$. The averaging is over an interval $[\varepsilon'' - \delta\varepsilon, \varepsilon'' + \delta\varepsilon]$ where $\delta\varepsilon = 150$. The linear dependence on $k = \sqrt{\varepsilon}$ is quite obvious in both cases. The mild fluctuations at higher energies are the remnants of the oscillatory off-diagonal terms in equation (2.8) which are washed out by the averaging procedure.

higher energies are due to the remnants of the oscillatory off-diagonal terms. These in fact vanish if the interval of averaging is increased. The higher slope in the first case is due to the smaller value of α.

Equation (3.5) thus provides a signature that is universal for integrable systems *irrespective of the nature of degeneracies in the orbit actions*. A few remarks are, however, in order. Isolated orbits do occur in integrable systems as well. The periodic orbit joining the mid-points of the sides of an equilateral triangle billiard is one such example. In general these arise in systems with discrete symmetry ($\boldsymbol{q}' = P_i\boldsymbol{q}$) when a single or a combination of a few parity classes are considered. For example, in systems with axes inversion the eigenstates can be classified under four parity classes and the respective Green function takes the form

$$G_{++} = (a_1 + a_2 + a_3 + a_4) \qquad G_{+-} = (a_1 + a_2 - a_3 - a_4)$$

$$G_{-+} = (a_1 - a_2 + a_3 - a_4) \qquad G_{--} = (a_1 - a_2 - a_3 + a_4)$$

where

$$a_1 = G(x, y; x, y) \qquad a_2 = G(-x, y; x, y)$$

$$a_3 = G(x, -y; x, y) \qquad a_4 = G(-x, -y; x, y).$$

In order to calculate the contribution to $G(P_i\boldsymbol{q}, \boldsymbol{q}; E)$, only classical trajectories located on or near a part of the periodic trajectory passing through the phase space points $(\boldsymbol{q}, \boldsymbol{p})$ and $(P_i\boldsymbol{q}, P_i^{-1}\boldsymbol{p})$ need be considered. Thus at times a single (isolated) orbit occurring in a one-parameter family may satisfy this condition.

The occurrence of these isolated orbits however, has little effect on the linear dependence of $\bar{\Delta}_3(\infty)$ on $\sqrt{\varepsilon}$. We have verified this in case of the equilateral triangle billiard. On the other hand, the occurrence of an $(N-1)$-parameter family of periodic orbits in chaotic systems affects the saturation value of the spectral rigidity and also its energy dependence. This can be seen in the Bunimovich stadium billiard where all periodic orbits are isolated with the exception of the bouncing ball modes (periodic oscillations between the straight edges). The latter variety contributes an amount equal to

$$(2m/\hbar^2)^{1/2} a^2 \zeta(3) E^{1/2} / 8\pi^3 b$$

to the spectral rigidity. Here 'a' is the length of the parallel sides and 'b' is their separation. The quantity $\zeta(3)$ is equal to $\Sigma(1/n^3)$. In comparison with the contribution of the isolated orbits, this is quite large and moreover dominates the saturation at higher energies.

We have computed $\bar{\Delta}_3(\infty)$ for the odd-odd parity mode of the stadium billiard at $L = 100$ and with $\delta\varepsilon = 150$. The saturation value in the range $\varepsilon = (345, 370)$ varies between 0.52 and 0.53 with minor fluctuations in between. Part of this comes from the bouncing ball modes which for this parity mode is equal to

$$(2m/\hbar^2)^{1/2} a^2 \zeta(3) E^{1/2} / 16\pi^3 b.$$

Its contribution therefore varies between 0.230 and 0.238, the remainder being due to the isolated orbits. Moreover, the increase in the value of $\bar{\Delta}_3(\infty)$, is due mostly to these bouncing ball modes.

Thus, the presence of $(N-1)$-parameter family periodic orbits in chaotic systems affects the saturation value and its energy dependence.

5. Conclusions

In the preceding sections, we have made a detailed study of the spectral rigidity for integrable systems with a view towards understanding the deviations from the universal behaviour $(L/15)$ in systems with degeneracies in orbit actions. In the process, we have also investigated the saturation region and shown that the energy dependence of $\bar{\Delta}_3(\infty)$ is a better indicator of the underlying universality in integrable systems.

Our conclusions can therefore be summarized as follows.

(i) We have shown that the diagonal approximation to $\phi(T)$ is inadequate for systems with degeneracies in orbit periods. The off-diagonal terms corresponding to degenerate orbits can however be incorporated in the diagonal sum $\phi_D(T)$. To a first approximation, $\bar{\Delta}_3(L) = n_{av}(L)\ L/15$ where $n_{av}(L)$ is the mean degeneracy of orbits which contribute to the spectral rigidity. Moreover the deviations are system specific and survive at high energies.

(ii) We find that the energy dependence of its saturation value is a better indicator of the underlying universality in generic integrable systems, where, $\bar{\Delta}_3(\infty) \sim \varepsilon^{1/2}$. On the other hand, the saturation value in chaotic systems with isolated periodic orbits, has a logarithmic dependence on energy.

Since terms identical or similar to $\phi(T)$ occur in the number variance, Σ^2, and the nearest-neighbour-level spacing distribution, $P(s)$, deviations in these spectral measures are also due to the degeneracies in orbit actions.

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